Green's function method for random fuse network problems

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We propose and study a method that self-consistently solves for the current and voltage distributions in random fuse network problems using lattice Green's functions. The method solves for the current distribution by only keeping track of the defect bonds and can treat networks with two *extreme* types of defects: insulating bonds and superconducting bonds. We apply this method to study the breakdown features of a simple two-dimensional superconductor-resistor-insulator network.

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I. INTRODUCTION

A variety of breakdown phenomena such as dielectric breakdown and brittle fracture have been modeled by networks with quenched randomness [1]. The scalar version of the problem is a random fuse network with a random distribution of elements with different I-V characteristics, between the nodes of a geometrically regular lattice. The dependence of the breakdown phenomenon, characterized by quantities such as the breakdown voltage and current and the topology of the "broken" bonds in the network, on the nature of the distribution, especially in the vicinity of a percolation threshold, has been of interest. The vector version of the problem is a random elastic network which has also been extensively investigated. Over the past few years, several numerical techniques and efficient algorithms have been developed to study network problems both close to and away from the percolation threshold [2–5]. The numerical problem for random fuse networks is to solve for node voltages using Kirchoff's law at each node and thereby obtain the current distribution. Some of the approaches use the well-known conjugate gradient method along with preconditioning techniques such as Fourier acceleration [6]. Other techniques to calculate the conductivity of fuse networks include bond-propagation [7] and transfer-matrix [8] methods. In this paper, we present a different method that directly solves for the current distribution in the random fuse network by keeping track of the defect bonds in contrast to the standard conjugate gradient method and its variants.

We illustrate the method by studying the breakdown of a simple two-dimensional fuse network on a square lattice with a fraction p_i of insulating bonds (infinite resistance) and p_s of shorted bonds (zero resistance). The motivation for the model comes partly from experimental investigations of several granular superconducting materials and disordered thin films which can be modeled by such networks [9]. A recent experiment by Yagil *et al.* [10] also finds evidence for the possible existence of these two kinds of defects in their gold (Au) and silver (Ag) films. While it may be difficult to generalize some of the existing methods to networks with both these two extreme types of defects present simultaneously, we show that our method can be applied to these problems. Furthermore, it has the advantage that the size of the matrix that enters the problem scales as $N \times N$, where N is the number of defect bonds in the network. Thus our method is especially suited to the study of models with low defect concentrations; this regime is also of relevance to certain polymer cracking models [11]. Near the percolation threshold we employ the well-known trick of removing the disconnected parts and the dead arms of the percolation cluster [12], thereby reducing the number of defect bonds; this makes our method numerically feasible [13].

We briefly summarize our results: It is useful to recall the important result derived by Duxbury *et al.* [2] that the breakdown properties of a fuse network in the presence of insulating defects depend crucially on the "most critical defect," i.e., the arrangement of a given number of insulating defects leading to the largest enhancement in the current through a fuse in the network; this led to the breakdown current scaling to zero logarithmically with the size of the system. We find similar results for the fuse networks with two types of defects. In particular, the behavior predicted by a Lifshitz type of argument, which is based on the most critical defect in the system for dilute defect concentrations, persists even in the presence of large variations in the resistances. In addition, for a given system size and defect concentrations, the distribution of the breakdown currents (for different defect configurations) is better described by a double exponential form similar to that in Refs. [2, 3] than the commonly used Weibull form [18]. We have also considered both avalanche dynamics and the one-at-a-time dynamics used in Ref. [2] while studying the breakdown of the entire network. Both dynamics yield qualitatively similar results.

II. MODEL AND METHODOLOGY

Our model consists of a square lattice network of size $L \times L$, as shown in Fig. 1, with each bond between neighboring sites representing either a fuse or a defect. The fuse behaves Ohmically with a resistance R up to a breakdown voltage and then it irreversibly becomes an

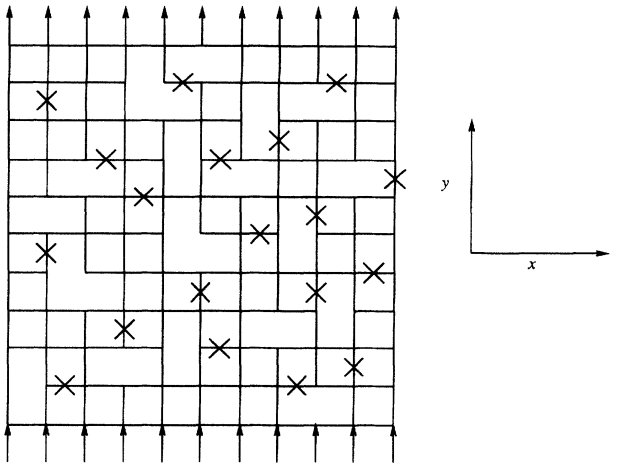

FIG. 1. A 10×10 random fuse network with free boundary conditions on the right and left edges. On the top and bottom edges, either constant current or constant voltage is applied. The defect concentrations p_i and p_s are approximately 0.1. The empty and crossed bonds represent insulating (type I defect) and shorted (type II defect) bonds, respectively.

insulator (a defect with infinite resistance). The defects in our model are bonds whose resistance R_d is not equal to R. We will focus on the following two types of defects in this paper: "type I" defects that are insulating, $R_d = \infty$, and "type II" defects that are superconducting, $R_d = 0$. Generalization to other defects is straightforward.

As the external current or voltage is increased some of the individual fuses will break until eventually there is a breakdown of the entire network, i.e., there is no conducting path across the network. The sequence of fuses that break can be determined as the external current is increased by finding the current distribution in the network for given external current or voltage. Clearly the fuse that breaks first is the one with the maximum current which depends on the initial defect distribution. Note also that as more fuses break, the number of type I defects increases and the current distribution has to be recalculated. Therefore we need a method to solve for the current and voltage distribution when the defect configuration and the boundary conditions are prescribed.

Let us first calculate the current distribution due to a single defect in an infinite system. Since the problem of determining the current distribution is *linear*, the solution of the problem with many defects is given by a linear combination of single-defect solutions. The key is to determine the coefficients of the linear combination. This can be accomplished, as we will show below, by solving a set of self-consistency equations.

A. Single defect

Let $I_x(\mathbf{r})$ and $I_y(\mathbf{r})$ denote the currents along the positive x and y directions, respectively, and $V(\mathbf{r})$ the voltage at site $\mathbf{r}$. The current distribution obeys Kirchoff's law, which enforces local current conservation at every node in the circuit:

$$I_x(\mathbf{r}) - I_x(\mathbf{r}-\hat{\mathbf{e}}_x) + I_y(\mathbf{r}) - I_y(\mathbf{r}-\hat{\mathbf{e}}_y) = 0. \tag{1}$$

The current in each bond in turn is related to the voltage drop by Ohm's law:

$$I_{x,y}(\mathbf{r}) = [V(\mathbf{r}+\hat{\mathbf{e}}_{x,y}) - V(\mathbf{r})]/R \tag{2}$$

at a normal bond and

$$I_{x,y}(\mathbf{r}) = [V(\mathbf{r}+\hat{\mathbf{e}}_{x,y}) - V(\mathbf{r})]/R_d \tag{3}$$

at a defect bond. Note that for a type I defect this corresponds to zero current; for a type II (superconducting) defect one has a zero voltage drop condition.

Consider now a single defect at the bond $(\mathbf{r}_0, \mathbf{r}_0 + \hat{\mathbf{e}}_x)$. To solve the problem we separate out the additional current and voltage contributions due to the presence of the defect:

$$I_{x,y} = I^{(0)}_{x,y} + I'_{x,y}, \tag{4}$$

$$V = V^{(0)} + V'. \tag{5}$$

In the preceding, $I^{(0)}$ ($V^{(0)}$) is the current (voltage) distribution in the absence of the defect and I' (V') is the additional current (voltage) due to the defect. The determination of I' is facilitated by separating it into two parts: $I'_x(\mathbf{r}) = I'^{(0)}_x(\mathbf{r}) + I_d\delta_{\mathbf{r},\mathbf{r}_0}$ and $I'_y(\mathbf{r}) = I'^{(0)}_y(\mathbf{r})$. We have defined $I'^{(0)}_{x,y}(\mathbf{r}) \equiv [V'(\mathbf{r}+\hat{\mathbf{e}}_{x,y}) - V'(\mathbf{r})]/R$; the contribution to the extra current is determined by the extra drop in voltage across each bond assuming that the defect bond also has a resistance R. The second term is a non-Ohmic or defect current, which is nonzero only at the defect bond and is necessary to satisfy Eq. (3). Using Kirchoff's law for I' (since I and I^0 satisfy Kirchoff's law so does I'), we obtain the equation for V':

$$V'(\mathbf{r}+\hat{\mathbf{e}}_x) + V'(\mathbf{r}-\hat{\mathbf{e}}_x) + V'(\mathbf{r}+\hat{\mathbf{e}}_y) + V'(\mathbf{r}-\hat{\mathbf{e}}_y)$$

$$-4V'(\mathbf{r}) = I_dR(\delta_{\mathbf{r},\mathbf{r}_0+\hat{\mathbf{e}}_x} - \delta_{\mathbf{r},\mathbf{r}_0}). \tag{6}$$

This is simply Poisson's equation for the potential on the lattice with a dipole source at the defect bond. Given the source I_d we can solve for V' for an infinite system using the lattice Green's function G_0 and the result is

$$\begin{aligned} V'(\mathbf{r}) &= I_dR[G_0(\mathbf{r}-\mathbf{r}_0) - G_0(\mathbf{r}-\mathbf{r}_0-\hat{\mathbf{e}}_x)] \\ &\equiv I_dRG_x(\mathbf{r}-\mathbf{r}_0), \end{aligned} \tag{7}$$

where G_0 [14] is given by

$$G_0(\mathbf{r}) = \frac{1}{(2\pi)^2} \times \int_{-\pi}^{\pi} dk_x \int_{-\pi}^{\pi} dk_y \frac{e^{i\mathbf{k}\cdot\mathbf{r}}}{4 - 2\cos(k_x) - 2\cos(k_y)}. \tag{8}$$

The solution for I' is easily seen to be

$$I'_x(\mathbf{r}) = I_d[G_x(\mathbf{r}-\mathbf{r}_0+\hat{\mathbf{e}}_x) - G_x(\mathbf{r}-\mathbf{r}_0)] + I_d\delta_{\mathbf{r},\mathbf{r}_0}, \tag{9}$$

$$I'_y(\mathbf{r}) = I_d[G_x(\mathbf{r}-\mathbf{r}_0+\hat{\mathbf{e}}_y) - G_x(\mathbf{r}-\mathbf{r}_0)]. \tag{10}$$

If the defect is along $\hat{\mathbf{e}}_y$, the corresponding results can be

obtained by replacing G_x by $G_y(\mathbf{r}) \equiv G_0(\mathbf{r}) - G_0(\mathbf{r}-\hat{\mathbf{e}}_y)$. The unknown defect current I_d can now be obtained self-consistently by choosing its value to satisfy Ohm's law across the defect bond [Eq. (3)]: $I_x(\mathbf{r}_0) = [V(\mathbf{r}_0+\hat{\mathbf{e}}_x) - V(\mathbf{r}_0)]/R_d = (I_x^{(0)} + I_x'^{(0)})R/R_d$; this leads to

$$I_x^{(0)} + I_d[G_x(\hat{\mathbf{e}}_x) - G_x(0)] + I_d = \{I_x^{(0)} + I_d[G_x(\hat{\mathbf{e}}_x) - G_x(0)]\}R/R_d. \quad (11)$$

Using the fact that $G_x(\mathbf{e}_x) - G_x(0) = -1/2$, we have

$$I_d = \frac{2(R-R_d)}{R+R_d} I_x^{(0)}. \quad (12)$$

Thus Eqs. (9), (10), and (12) provide the solution to the problem of a single defect in an infinite fuse network.

B. Many defects

For the problem with many defects in a finite system, we can similarly characterize each defect or boundary site with a "defect current" I_d. By solving a set of self-consistent equations which are the generalization of Eq. (11) for a single defect, we can obtain I_d for all defect and boundary bonds; these defect currents determine the current distribution in the entire system.

The detailed procedure is outlined below: As stated earlier, we restrict our attention to two types of defect bonds: type I bonds with $R_d = \infty$ and type II bonds with $R_d = 0$. We emphasize, however, that the method we present below can be applied to study a set of defect bonds with any distribution of defect resistances. Suppose there are N defect bonds at the locations $\{\mathbf{r}_n\}$; the direction of the bond is denoted by i_n, which can be either x or y. The solution for the currents in the defect bonds given the defect currents I_{dn} at the nth defect bond, which are yet to be determined, can be written as

$$I_{i_n}(\mathbf{r}_n) = I_{i_n}^{(0)}(\mathbf{r}_n) + \sum_m I_{dm}[\, G_{i_m}(\mathbf{r}-\mathbf{r}_m+\hat{\mathbf{e}}_{i_n}) - G_{i_m}(\mathbf{r}-\mathbf{r}_m)] + I_{dn}. \quad (13)$$

This is the analog of Eqs. (9) and (10) restricted to defect bonds only and the sum is over all defect bonds that induce an Ohmic contribution at the nth bond.

The values of the defect currents are determined as before by imposing Ohm's law whose form depends on the type of the defect: For an insulating type I bond the condition is $I_{i_n}(\mathbf{r}_n) = 0$ and for a superconducting type II bond, $I_{i_n}(\mathbf{r}_n) = I_{dn}$ [this corresponds to a zero voltage drop or equivalently $I_{i_n}^{(0)}(\mathbf{r}_n) + I_{i_n}'^{(0)}(\mathbf{r}_n) = 0$]. For convenience let us assume that the first N_1 bonds are type I bonds, while the rest are type II bonds. Thus for $n \le N_1$ (type I bonds) we have

$$I_{i_n}(\mathbf{r}_n) = I_{i_n}^{(0)}(\mathbf{r}_n) + \sum_m I_{dm}[\, G_{i_m}(\mathbf{r}-\mathbf{r}_m+\hat{\mathbf{e}}_{i_n}) - G_{i_m}(\mathbf{r}-\mathbf{r}_m)] + I_{dn} = 0; \quad (14)$$

for $n > N_1$ (type II bonds) we have

$$I_{i_n}(\mathbf{r}_n) - I_{dn} = I_{i_n}^{(0)}(\mathbf{r}_n) + \sum_m I_{dm}[\, G_{i_m}(\mathbf{r}-\mathbf{r}_m+\hat{\mathbf{e}}_{i_n}) - G_{i_m}(\mathbf{r}-\mathbf{r}_m)] = 0. \quad (15)$$

Equations (14) and (15) are linear equations and must be solved to determine the set of defect currents $\{I_{dn}\}$.

Once the set of defect currents are obtained, the currents in all the bonds can be calculated using the following equations that are analogous to Eqs. (9) and (10) in the single defect case:

$$I_x(\mathbf{r}) = I_x^0(\mathbf{r}) + \sum_n I_{dn}[\, G_{i_n}(\mathbf{r}-\mathbf{r}_n+\hat{\mathbf{e}}_x) - G_{i_n}(\mathbf{r}-\mathbf{r}_n)] + \sum_n I_{dn}\delta_{\mathbf{r},\mathbf{r}_n}\delta_{x,i_n}, \quad (16)$$

$$I_y(\mathbf{r}) = I_y^0(\mathbf{r}) + \sum_n I_{dn}[\, G_{i_n}(\mathbf{r}-\mathbf{r}_n+\hat{\mathbf{e}}_y) - G_{i_n}(\mathbf{r}-\mathbf{r}_n)] + \sum_n I_{dn}\delta_{\mathbf{r},\mathbf{r}_n}\delta_{y,i_n}. \quad (17)$$

To find the defect currents ($\{I_{dn}\}$) from the self-consistency conditions, we can use the following two methods.

Method 1

Equations (14) and (15) can be written as a matrix equation:

$$A_1 C = C_0, \quad (18)$$

where C_0 is a vector with components $C_0(n) = I_{i_n}^{(0)}(\mathbf{r}_n)$ for $n = 1,\ldots,N$ and C is a vector whose components are the unknown defect currents: $C(n) = I_{dn}$ for $n = 1,\ldots,N$. A_1 is a known symmetric matrix with elements related to the Green's functions. The solution of this equation can be obtained using a standard linear equation routine.

Method 2

Equations (14) and (15) can also be solved by iterative schemes. We illustrate the method here by using a straightforward procedure. More efficient over-relaxation techniques can be devised to speed up the algorithm. The iteration equations that determine the currents at step $t+1$ given those at t can be written as follows:

$$I_n(t+1) = I_n(t) + \sum_m \tilde{I}_m(t)[\, G_{i_m}(\mathbf{r}_n-\mathbf{r}_m+\hat{\mathbf{e}}_{i_n}) - G_{i_m}(\mathbf{r}_n-\mathbf{r}_m)] + \tilde{I}_n(t) \quad (19)$$

for $n \le N_1$ and

$$I_n(t+1) = I_n(t) + \sum_m \tilde{I}_m(t)[\, G_{i_m}(\mathbf{r}_n-\mathbf{r}_m+\hat{\mathbf{e}}_{i_n}) - G_{i_m}(\mathbf{r}_n-\mathbf{r}_m)] \quad (20)$$

for $n > N_1$. We set the initial ($t = 0$) values by $I_n(t=0) = I_{i_n}^{(0)}(\mathbf{r}_n)$ for all N defects. At $t = 0$ and after each iteration we set $\tilde{I}_n(t) = -2I_n(t)$ for the N_1

type I bonds and set $\tilde{I}_n(t) = 2I_n(t)$ for the N_2 type II bonds. Note that the above equations can be written as a matrix equation:

$$C(t+1) = A_2 C(t) \tag{21}$$

as in method 1, where A_2 is a traceless real matrix. We perform the iteration T times until the solution converges to a desired accuracy [i.e., $\|C_n(t)\| \leq \epsilon$ for each n corresponding to nth defect bond] [15]. It is easy to check that the effective defect bond currents are given by

$$I_{dn} = \sum_{t=0}^{T} \tilde{I}_n(t). \tag{22}$$

C. Boundary conditions

Consider an $L \times L$ square lattice. For the network in Fig. 1, (i.e., with free boundary conditions), there is no current flow through the left and right edges. This is ensured by placing insulating (type I) bonds at the right and left edges along the x direction. A constant input (output) current boundary condition along $\hat{\mathbf{e}}_y$ at bottom (top) boundaries (see Fig. 1) can be considered by placing modified type I defects at the bottom (top) y bonds [16]. A constant voltage bias also can be applied to the network along $\hat{\mathbf{e}}_y$ by placing type II defects at the top ($y = L$) and bottom ($y = 1$) x bonds.

III. RESULTS

We study the breakdown of the network with defect concentrations p_i of type I and p_s of type II. For simplicity we take the resistance and the breakdown voltage of a fuse to be unity. We envisage a situation in which the input current is increased gradually, slowly compared to the time scale on which the network reaches steady state for a given external current. Accordingly we introduce "avalanche dynamics" to describe the breakdown. In real materials the dynamics of the breakdown involves additional complexity including thermal effects and their time scales [17]. Hence the avalanche procedure we adopt here is a simplification. The avalanche is initiated when the voltage across a fuse exceeds unity and breaks (becomes an insulator) irreversibly. We now maintain the input current at a fixed value. The distribution of currents in the network obtained after the first fuse breakage can cause the voltage across other fuses to exceed unity resulting in further breakages. An avalanche consisting of a sequence of fuse breakages ensues but ends when no more fuses can break in the system and the network is still conducting. The input current is increased after the avalanche until a new avalanche is initiated and so on until the whole network breaks down. We have also performed the simulation using a different (one-at-a-time, i.e., sequential breakage of fuses with maximum current until the breakdown of the entire network) dynamics described in Ref. [2] and the results are qualitatively similar to the avalanche procedure described here.

We have studied $L \times L$ square lattices of sizes L up to 80 with both type I and type II defects using the method described above. We have also considered lattice sizes up to $L = 128$ for low defect concentrations. Typically our method takes 16–20 seconds for $L = 80$ system with defect concentrations $p_i = p_s = 0.1$ to converge to the solution with an accuracy 10^{-8} on the Cray YMP supercomputer.

We study, following Ref. [2], both $i_1 = I_1/L$, the input current per node at which the first fuse breaks in the network, and $i_b = I_b/L$, the input current per node at which the whole network breaks (there is no conducting path in the network). Above the percolation threshold (when the system has finite conductance), for $p_i = 0$ and $p_s \neq 0$, i.e., with finite concentration of the type II defects, one can argue that i_1 depends on the largest probable linear defect in the system, as in the case with purely type I defects considered in Ref. [2]. There is one difference, however, in the direction of the "most critical defect": The most critical defect in this case is along the direction of the input current in contrast to the case with purely type I defects in which the defect extends transverse to the input current direction. Nevertheless, the arguments presented in Ref. [2] for the case of finite

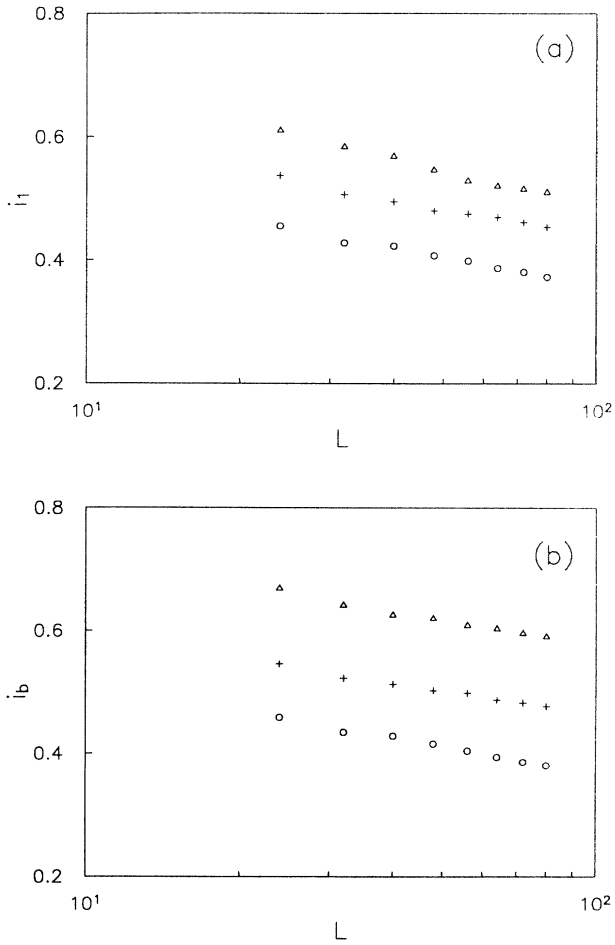

FIG. 2. The dependence of the average breakdown current (a) i_1 and (b) i_b on the size of the system L. The average is taken over 50 configurations for each L. The data are obtained with $p_i = 0.1, p_s = 0.0$ (circles), $p_i = 0.0, p_s = 0.1$ (triangles), $p_i = 0.05$, and $p_s = 0.05$ (crosses).

concentration of type I bonds remain valid for this linear type II defect. One can solve for the current density at the boundary of an elliptical type II defect in an infinite medium following the calculation for the type I elliptical defect in Ref. [2]. The maximum current density is given by $j_0(1 + b/a)$, where a and b are the semiaxis lengths transverse and parallel to the current flow, respectively, and j_0 is the current density at infinity. The maximum density occurs at the tips of the defect *along* the current flow direction in contrast to the case of a type I elliptical hole where the maximum value of $j_0(1 + a/b)$ occurs at the tips *transverse* to the current flow. Hence, following the Lifshitz type of argument one expects that, for small defect concentrations, the breakdown current i_1 vanishes logarithmically with the size of the system: $i_1 \approx \text{const}[1 + K(\ln L)^{\alpha}]^{-1}$ [2]. This is indeed observed in our simulation (Fig. 2) and our data are consistent with $\alpha \approx 1$. In addition, we find numerically that the logarithmic dependence of i_1 on the size of the system persists even for the case when both type I and type II defects are present in the network. This suggests that the Lifshitz type of argument remains valid for this case where the most critical defect is a combination of type I and type II defects.

What is less clear is the scaling behavior of i_b. In this case we must consider a mixture of type I and type II defects (even for the case of only type II defects in the initial configuration), because type I defects are continuously generated as the breakdown process proceeds. Naively one might expect a different scaling behavior from the case of purely type I defects as type II defects lead to a different current distribution. This turns out not to be the case. Our results show that the scaling behavior of i_b does not depend on the existence of type II defects. A careful consideration of the simulation process suggests the following explanation of this result: As the breakdown process continues, more and more type I defects are generated. On the other hand, type II defects are gradually "screened" (surrounded) by the type I defects; this occurs because the fuses neighboring a type II defect carry relatively higher currents and are more likely to break. The behavior of the breakdown process can be expected to be dominated by type I defects, and hence the scaling behavior for i_b should not be sensitive to the

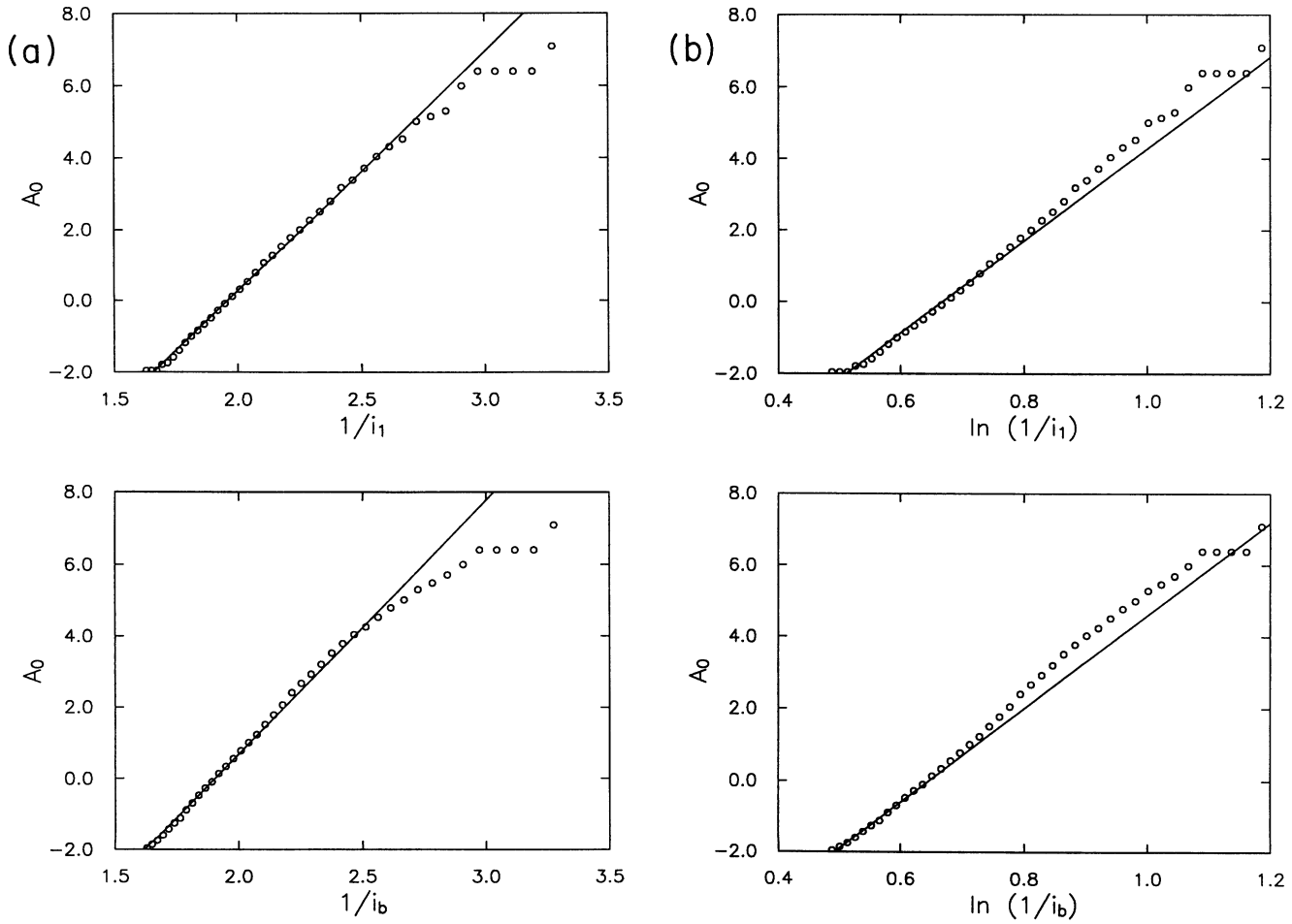

FIG. 3. Cumulative distributions F of the breakdown currents i_1 and i_b for a fixed defect concentration. A 40×40 system is used with $p_i = p_s = 0.05$. (a) The quantity $A_0 = -\ln\{-\ln[1 - F(i_{1,b})]\}$ is plotted vs $1/i_{1,b}$. For a double exponential distribution the plot should be linear. (b) A_0 is plotted vs $\ln(1/i_{1,b})$. For a Weibull distribution the plot should be linear.

existence of type II defects. The dependence of the average breakdown currents i_b on the size of the system L is shown in Fig. 2(b).

We have also obtained the cumulative distributions F of the breakdown currents (i_1, i_b) for a fixed defect concentration and size of the system $L = 40$ by taking the data for 2000 configurations. The double exponential form $F_{de}(i) = 1 - \exp[-c_1 \exp(-c_2/i)]$, suggested in Ref. [2], affords a better fit for these distributions (see Fig. 3) than the commonly used Weibull form [2, 18]: $F_w(i) = 1 - \exp[-c\, i^m]$ in breakdown phenomena. However, the double exponential fit is not as good for small i_b as that for i_1.

These results indicate that the breakdown behavior above the percolation threshold is unchanged in the presence of type II defects and the attendant large variations in the resistances.

IV. CONCLUSIONS

In conclusion we have devised a Green's function method to solve for the current distributions in random fuse network problems. This method crucially differs from the existing conjugate gradient methods as it only keeps track of the defect bonds to obtain the current distribution. It is easy to consider two different types of conjugate defects with zero and infinite resistance in this formulation. We have considered a network with both types of defects and find that the breakdown features are not altered in the presence of large variations in the defect bond resistances. This method can be useful in the study of disorder in systems such as high T_c superconducting films which can be modeled as a network with both kinds of defects. We finally remark that our method can be generalized to study crack generation in elastic materials, for example, polymer crack problems, in which a small fraction of bonds rupture during the elastic breakdown.

ACKNOWLEDGMENTS

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[14] Such Green's functions have been used in a model study of the dynamics of earthquakes by K. Chen, P. Bak, and S. P. Obukhov, Phys. Rev. A **43**, 625 (1991).

[15] The matrix A_2 has eigenvalues with magnitude between -1 and 1. In some cases (when defects in a disconnected cluster are included in the self-consistency procedure), A_2 may have eigenvalues equal to -1. In these cases, we have to either remove the disconnected cluster or use a modified iteration procedure.

[16] To consider constant input-output currents along $\hat{\mathbf{e}}_y$ at the top and bottom boundaries, we need to fix the current on the bonds at the top and bottom boundaries along $\hat{\mathbf{e}}_y$ to a given constant value. This can be accomplished by treating these bonds as modified type I defects, where the self-consistency condition requires the currents on these bonds to be equal to the given constant input current instead of zero as in the case of type I bonds discussed in the text. We have considered both constant current and constant voltage boundary conditions. By representing the boundary of the system using "defect currents" and choosing appropriate self-consistency conditions we can study a variety of boundary conditions using our method.

[17] Recently, a dynamical thermal fuse model has been studied by D. Sornette and C. Vanneste, Phys. Rev. Lett. **68**, 612 (1992).

[18] E. J. Gumbel, *Statistics of Extremes* (Columbia University Press, New York, 1958).